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MULTIDIMENSIONAL LAPLACE APPROXIMATION VIA TROTTER OPERATOR

Abstract. *The classical distribution of Laplace, along with the normal one, became one of the most actively used symmetric probabilistic models. A separate task of mathematics is the Laplace approximation, i.e. method of estimating the parameters of the normal distribution in the approximation of a given probability density. In this article the problem of Laplace approximation in d-dimensional space has been investigated. In particular, the rates of convergence in problems of the multidimensional Laplace approximation are studied. The mathematical tool used in this article is the operator method developed by Trotter. It is very elementary and elegant. Two theorems are proved for the evaluation of convergence rate. The convergence rates, proved in the theorems, are expressed using two different types of results, namely: estimates of the convergence rate of the approximation are obtained in terms of "large-O" and "small-o". The received results in this paper are extensions and generalizations of known results. The results obtained can be used when using the Laplace approximation in machine learning problems. The results in this note present a new approach to the Laplace approximation problems for the d-dimensional independent random variables.*
Keywords: *Laplace approximation; geometric sums; random sums; Trotter operator; the rates of convergence*

1. Introduction

The Laplace distribution was introduced by P. S. Laplace in 1774. It is also said to be the first law of errors. The Laplace distribution appears in a number of applications in the sciences, in business and in branches of engineering. Recently, the Laplace approximation problem has been used in information technology, in particular, in machine learning [1]. In recent years, the Laplace approximation problems have been interested by many mathematicians. However, the results only focus on 1-dimensional space. In this paper, we will solve the Laplace approximation problems on d-dimension space. Laplace distribution on d-dimensional space is defined as the following.

Let $R^d = \left\{ \mathrm{x} = \left(x_1, x_2, ..., x_d \right) \mid x_i \in R, i = 1, 2, ..., d \right\}$ be a d-dimensional Euclidean space with norm $\|x\| = \left(\sum_{i=1}^{d} x_i^2 \right)^{\frac{1}{2}}$. Let $\{X_n, n \geq 1\}$ be a sequence of d-dimensional random vectors and F_{X_n} be a the distribution function of X_n. Let Z be a d-dimensional Laplace random vector with characteristic function

$$\varphi_Z(\mathrm{t}) = \frac{e^{i\mathrm{m}'\mathrm{t}}}{1 - i\mathrm{a}'t + \frac{1}{2}\mathrm{t}'\Sigma\mathrm{t}},$$

where: $\mathrm{a}, \mathrm{m} \in R^d$ and $\boldsymbol{\Sigma}$ is a $d \times d$ symmetric and positive definite matrix and $\mathrm{m}', \mathrm{a}', \mathrm{t}'$ are the transposes of $\mathrm{m}, \mathrm{a}, \mathrm{t},$ respectively. We use the notation $L_d(\mathrm{m}, \mathrm{a}, \Sigma)$ to denote the distribution of Z, and write $Z \sim L_d(\mathrm{m}, \mathrm{a}, \Sigma)$. The distribution is also called to be multivariate asymmetric Laplace. Suppose that ν is a geometrically distributed random variable with parameter p $(0 < p < 1)$, in short, $\nu \sim Geo(p)$, and independent of the sequence $\{X_n, n \geq 1\}$. It is well known that under desired conditions,

$$p^{\frac{1}{2}} \sum_{j=1}^{\nu} \left(X_j + b_j \right) \xrightarrow{d} Z \quad as \quad p \to 0, \quad (1)$$

where: $\xrightarrow{d}$ is meaning the convergence in distribution. If $b_j = \mathrm{a}\left(p^{\frac{1}{2}} - 1 \right)$, $Z \sim L_d(0, \mathrm{a}, \Sigma)$, where: a, Σ are mean and covariance matrix of X_j, respectively (see [2], for more details). Furthermore, if $b_j = p^{\frac{1}{2}} a_j$, then $Z \sim L_d(0, \mathrm{a}, \Sigma)$, where: $a = \lim_{n \to +\infty} \frac{1}{n} a_j$ and $\boldsymbol{\Sigma} = \lim_{n \to +\infty} \sum_{j=1}^{n} Var\left[X_j \right]$ (see [3], for more details).

In 2014, the authors of Hung and Giang used the Trotter-Renyi method to solve the Poisson approximation problems (see [4], for more details). The Trotter-Renyi method is a special case of the Trotter method and is only used for discrete random variables. However, to solve the Laplace approximation problems we must use the Trotter method. We will learn about this method in the next section.

This paper is organized as follows. We start in Section 2 by reviewing of Trotter operator and their properties. The class of continuous modulus and Lipschitz functions is utilized in the paper. In section 3, we give main results of this paper. Conclusions of this study are presented in the last section.

2. Materials and Methods

Definition 1. $T_X : C_B(R^d) \to C_B(R^d)$ is defined by

$$T_X f(\mathrm{y}) = \int_{R^d} f(\mathrm{x}+\mathrm{y})\, dF_X(\mathrm{x}),$$

where: $f \in C_B(R^d)$, $C_B(R^d)$ is the set of all real–valued, bounded, uniformly continuous functions defined on the R^d and X is a random vector.

The properties of Trotter operator can be seen in [5], [6], [7] and [8]. Before starting the main results of this paper we review the properties of Trotter operator.

1. Every $f \in C_B(R^d)$, we have

$$\|T_X f\| < \|f\|$$

with $\|f\| = \sup\{|f(\mathrm{y})| : \mathrm{y} \in R^d\}$

2. T_X is a linear operator.

3. If X_1, X_2 are identically distributed, $T_{X_1} f \equiv T_{X_2} f, \quad \forall f \in C_B(R^d)$.

4. Suppose that X_1, X_2 are independent random vectors with distribution functions F_1, F_2 , then

$$T_{X_1+X_2} f \equiv (T_{X_1} \circ T_{X_2}) f \equiv (T_{X_2} \circ T_{X_1}) f, \quad \forall f \in C_B(R^d)$$

5. Now assuming that $X_1, X_2, ..., X_n$ are independent random vectors with distribution functions $F_1, F_2, ..., F_n$. Then, we get

$$T_{X_1+X_2+...+X_n} f \equiv (T_{X_1} \circ T_{X_2} \circ ... \circ T_{X_n}) f, \quad \forall f \in C_B(R^d).$$

6. If $X_1, X_2, ..., X_n$ and $W_1, W_2, ..., W_n$ are independent random vectors, then for each $f \in C_B(R^d)$, we have

$$\left\| T_{\sum_{i=1}^{n} X_i} f - T_{\sum_{i=1}^{n} W_i} f \right\| \le \sum_{i=1}^{n} \left\| T_{X_i} f - T_{W_i} f \right\|.$$

7. Assuming that there are sequences of random vectors $X_1, X_2, ..., X_n$ and $W_1, W_2, ..., W_n$ independent and independent with positive-valued random variables N. Then for each $f \in C_B(R^d)$, we get

$$\left\| T_{\sum_{i=1}^{N} X_i} f - T_{\sum_{i=1}^{N} W_i} f \right\| \le \sum_{n=1}^{\infty} P(N=n) \left\| T_{\sum_{i=1}^{n} X_i} f - T_{\sum_{i=1}^{n} W_i} f \right\|.$$

8. A sufficient condition for $X_n \Rightarrow X \quad as \quad n \to \infty$ is

$$\|T_{X_n} f - T_X f\| \to 0, \quad as \quad n \to \infty,$$

for each $f \in C_B^s(R^d), s \ge 1$.

We need to recall the definition of the modulus of continuity and Lipchitz classes.

Definition 2. If $f \in C_B(R^d), \mathrm{x}, \mathrm{h} \in R^d, \delta > 0$, then function

$$\omega(f;\delta) := \sup_{\|\mathrm{h}\| \le \delta} |f(\mathrm{x+h}) - f(\mathrm{x})|,$$

is called the modulus of continuity of function f .

The basic properties of this function mentioned following:

1) It grows monotonically with respect to δ ;

2) $\omega(f;\delta) \to 0$ as $\delta \to 0^+$;

3) $\omega(f;\lambda\delta) \le (1+\lambda)\omega(f;\delta)$,

where: $\lambda \in R$.

We say that the function $f \in C_B(R^d)$ satisfies a Lipchitz condition with exponent α $(0 < \alpha < 1)$, we write $f \in Lip(\alpha)$, if

$$\omega(f;\delta) = O(\delta^\alpha).$$

For any $f \in C_B^s(R^d), \delta > 0,$ define $\omega(s; f;\delta) = sup\{\omega(g;\delta) : g$ is any partial derivative of order $j (0 \le j \le s)$ of $f\}$.

It is easy to see that, for any $\lambda > 0$, we have

$$\omega(s; f;\lambda\delta) \le (1+\lambda)\omega(s; f;\delta).$$

Lemma 1. If $f \in C_B^s(R^d), \mathrm{x}, \mathrm{y} \in R^d$, then

$$f(x+y) = \sum_{j=0}^{s} \frac{(x.\nabla)^j}{j!} f(y) + \frac{1}{s!}(x.\nabla)^s \left[f(\eta) - f(y) \right],$$

where: η is such that $\|\eta - \mathrm{y}\| \le \|\mathrm{x}\|$

and $(x.\nabla) = \sum_{k=1}^{d} x_k . \frac{\partial f}{\partial x_k}$.

Proof of this Lemma can be found in ([9], p. 277).

3. Results

Lemma 2. Assuming $Z \sim L_d\left(0, \mathrm{a}, \Sigma\right)$, then

$$Z \overset{d}{=} p^{\frac{1}{2}} \sum_{i=1}^{\upsilon} Z_i,$$

where: $Z_i \sim L_d\left(0; p^{\frac{1}{2}}\mathrm{a}; \Sigma\right), \upsilon \sim Geo(p)$ and independent of $\{Z_i, i \ge 1\}$.

Proof. Each $\mathrm{t} \in R^d$, we get

$$\varphi_{p^{\frac{1}{2}}\sum_{i=1}^{\upsilon} Z_i}(\mathrm{t}) = \psi_\upsilon \circ \varphi_{Z_1}(p^{\frac{1}{2}}\mathrm{t})$$

$$= \frac{p.\varphi_{Z_1}(p^{\frac{1}{2}}\mathrm{t})}{1-(1-p).\varphi_{Z_1}(p^{\frac{1}{2}}\mathrm{t})}$$

$$= \frac{1}{1 - i\mathrm{a}'\mathrm{t} + \frac{1}{2}\mathrm{t}'\Sigma\mathrm{t}} = \varphi_Z(\mathrm{t}).$$

The proof of this lemma is completed.

Lemma 3. If $f \in C_B^s(R^d), \mathrm{x}, \mathrm{y}, \eta \in R^d, s \in N$, then the following inequalities are held

i) $\left|(x\nabla)^s\left[f(\eta) - f(y)\right]\right| \le d^{\frac{s}{2}}\|x\|^s\omega\left(s; f; \delta\right)$, where: $\|\eta - y\| \le \delta.$

ii) $\left|(x\nabla)^s\left[f(\eta) - f(y)\right]\right| \le 2d^{\frac{s}{2}}\|x\|^s\left\|f^{(s)}\right\|$, where $\left\|f^{(s)}\right\| = sup\{\|g\|$, g is any partial derivative of order s of $f\}$.

iii) $\forall \varepsilon > 0, \exists \delta > 0$ such as $\|\eta - y\| \le \delta$, then $\left|(x\nabla)^s\left[f(\eta) - f(y)\right]\right| \le \varepsilon d^{\frac{s}{2}}\|x\|^s$.

Proof

i) On account of the properties of $\omega(s-1; f; \delta)$,

$$\left|(x\nabla)^s\left[f(\eta) - f(y)\right]\right|$$

$$= \left|\sum_{k_1, k_2, \dots, k_d} \frac{s!}{k_1!k_2!\dots k_d!} x_1^{k_1}x_2^{k_2}\dots x_d^{k_d}\left[A\right]\right|$$

$$\le \omega\left(s; f; \delta\right)\left|\sum_{i=1}^{d} x_i\right|^s \le d^{\frac{s}{2}}\|x\|^s\omega\left(s; f; \delta\right),$$

where: $A = \frac{\partial^s f(\eta)}{\partial x_1^{k_1}\partial x_2^{k_2}\dots\partial x_d^{k_d}} - \frac{\partial^s f(y)}{\partial x_1^{k_1}\partial x_2^{k_2}\dots\partial x_d^{k_d}}$.

ii) Since $\left\|f^{(s)}\right\| = sup\left\{\|g\| : g\right.$ is any partial derivative of order s of $f\}$, it is easy to get

$$\left|(x\nabla)^s\left[f(\eta) - f(y)\right]\right|$$

$$= \left|\sum_{k_1, k_2, \dots, k_d} \frac{s!}{k_1!k_2!\dots k_d!} x_1^{k_1}x_2^{k_2}\dots x_d^{k_d}\left[A\right]\right|$$

$$\le \left\|f^{(s)}\right\|\left|\sum_{i=1}^{d} x_i\right|^s \le d^{\frac{s}{2}}\|x\|^s\left\|f^{(s)}\right\|.$$

where: $A = \frac{\partial^s f(\eta)}{\partial x_1^{k_1}\partial x_2^{k_2}\dots\partial x_d^{k_d}} - \frac{\partial^s f(y)}{\partial x_1^{k_1}\partial x_2^{k_2}\dots\partial x_d^{k_d}}$.

iii) Since $f \in C_B^s(R^d)$, $\forall \varepsilon > 0, \exists \delta > 0$ such as $\|\eta - y\| \le \delta$, then

$$\left|(x\nabla)^s\left[f(\eta) - f(y)\right]\right|$$

$$= \left|\sum_{k_1, k_2, \dots, k_d} \frac{s!}{k_1!k_2!\dots k_d!} x_1^{k_1}x_2^{k_2}\dots x_d^{k_d}\left[A\right]\right|$$

$$\le \varepsilon\left|\sum_{i=1}^{d} x_i\right|^s \le \varepsilon d^{\frac{s}{2}}\|x\|^s.$$

where: $A = \frac{\partial^s f(\eta)}{\partial x_1^{k_1}\partial x_2^{k_2}\dots\partial x_d^{k_d}} - \frac{\partial^s f(y)}{\partial x_1^{k_1}\partial x_2^{k_2}\dots\partial x_d^{k_d}}$.

Theorem 1. Assuming that $\{X_n, n \ge 1\}$ is a sequence of d-dimensional independent random variables for which

$$\vartheta_i(j) := \sum_{\substack{1 \le i_1, i_2, \dots, i_j \le d \\ s_1 + s_2 + \dots + s_j = j}} \left|\int_{R^d} B\right| = 0, \qquad (2)$$

where: $B = x_{i_1}^{s_1}x_{i_2}^{s_2}\dots x_{i_j}^{s_j} d\left[F_{X_i}(x) - F_{Z_i}(x)\right]$

and

$$\vartheta_{i,s} := \int_{R^d} \|x\|^s d\left[F_{X_i}(x) + F_{Z_i}(x)\right] < +\infty \qquad (3)$$

where: $s \ge 3$ is a fixed integer, $1 \le j < s, i = 1, 2, \dots, n$. Then, for any $f \in C_B^{s-1}(R^d)$,

$$\| T_{S_\upsilon} f - T_Z f \| \le 2\frac{d^{\frac{s-1}{2}}}{(s-1)!} p^{\frac{s-1}{2}} WE,$$

where: $WE = \omega\left(s-1; f; p^{\frac{1}{2}}\right) E\left[\sum_{i=1}^{\upsilon}\left(1+\vartheta_{i,s}\right)\right]$

$$S_\upsilon = p^{\frac{1}{2}} \sum_{i=1}^{\upsilon} X_i .$$

In addition, if $f^{(s-1)} \in Lip(\alpha)$, then

$$\| T_{S_\upsilon} f - T_Z f \| = O\left\{ p^{\frac{s-1+\alpha}{2}} E\left[\sum_{i=1}^{\upsilon}\left(1+\vartheta_{i,s}\right)\right]\right\}.$$

Furthermore, if X_n are independent and identically distributed d-dimensional random variables, then

$$\| T_{S_\upsilon} f - T_Z f \| = O\left(p^{\frac{s-3+\alpha}{2}}\right).$$

Proof. Since $f \in C_B^{s-1}(R^d)$, by virtue of lemma 1, we have

$$f\left(p^{\frac{1}{2}}x + y\right) = \sum_{j=0}^{s-1} \frac{\left(p^{\frac{1}{2}}\right)^j}{j!} (x.\nabla)^j f(y) + \frac{\left(p^{\frac{1}{2}}\right)^{s-1}}{(s-1)!} (x.\nabla)^{s-1}\left[f(\eta) - f(y)\right],$$

where: $\|\eta - y\| \le p^{\frac{1}{2}}\|x\|$. Hence

$$T_{p^{\frac{1}{2}}X_i} f(y) = \int_{R^d} \left(p^{\frac{1}{2}}x + y\right) dF_{X_i}(x) = \sum_{i=0}^{s-1} \frac{\left(p^{\frac{1}{2}}\right)^j}{j!} \int_{R^d} (x\nabla)^j f(y) dF_{X_i}(x) + \frac{\left(p^{\frac{1}{2}}\right)^{s-1}}{(s-1)!} \int_{R^d} (x\nabla)^{s-1}\left[f(\eta_1) - f(y)\right] dF_{X_i}(x) \tag{4}$$

where: $\|\eta_1 - y\| \le p^{\frac{1}{2}}\|x\|$. On the other hand, the similar arguments give us

$$T_{p^{\frac{1}{2}}X_i} f(y) = \int_{R^d} \left(p^{\frac{1}{2}}x + y\right) dF_{Z_i}(x) = \sum_{i=0}^{s-1} \frac{\left(p^{\frac{1}{2}}\right)^j}{j!} \int_{R^d} (x\nabla)^j f(y) dF_{Z_i}(x) + \frac{\left(p^{\frac{1}{2}}\right)^{s-1}}{(s-1)!} \int_{R^d} (x\nabla)^{s-1}\left[f(\eta_2) - f(y)\right] dF_{Z_i}(x) \tag{5}$$

where: $\|\eta_2 - y\| \le p^{\frac{1}{2}}\|x\|$. On account of (2), (3), combining (7), (8) we have

$$\left| T_{p^{\frac{1}{2}}X_i} f(y) - T_{p^{\frac{1}{2}}Z_i} f(y) \right| \le \frac{\left(p^{\frac{1}{2}}\right)^{s-1}}{(s-1)!}\{C + D\},$$

$$C = \int_{R^d} \left| (x\nabla)^{r-1}\left[f(\eta_1) - f(y)\right] \right| dF_{X_i}(x),$$

$$D = \int_{R^d} \left| (x\nabla)^{s-1}\left[f(\eta_2) - f(y)\right] \right| dF_{Z_i}(x).$$

On account of the properties of $\omega(s-1; f; \delta)$ and lemma 3, we have

$$\left| (x\nabla)^{s-1}\left[f(\eta_1) - f(y)\right] \right| \le d^{\frac{s-1}{2}}\left(\|x\|^{s-1} + \|x\|^s\right)\omega\left(s-1; f; p^{\frac{1}{2}}\right),$$

the similar arguments give us

$$\left| (x\nabla)^{s-1}\left[f(\eta_2) - f(y)\right] \right| \le d^{\frac{s-1}{2}}\left(\|x\|^{s-1} + \|x\|^s\right)\omega\left(s-1; f; p^{\frac{1}{2}}\right).$$

Therefore,

$$\left| T_{p^{\frac{1}{2}}X_i} f(y) - T_{p^{\frac{1}{2}}Z_i} f(y) \right| \le \frac{\left(p^{\frac{1}{2}}\right)^{s-1}}{(s-1)!} d^{\frac{s-1}{2}} \omega\left(s-1; f; p^{\frac{1}{2}}\right) \int_{R^d}\left(\|x\|^{s-1} + \|x\|^s\right) FF$$
$$= \frac{\left(p^{\frac{1}{2}}\right)^{s-1}}{(s-1)!} d^{\frac{s-1}{2}} \omega\left(s-1; f; p^{\frac{1}{2}}\right)\left(\vartheta_{i,s-1} + \vartheta_{i,s}\right)$$
$$\le 2\frac{\left(p^{\frac{1}{2}}\right)^{s-1}}{(s-1)!} d^{\frac{s-1}{2}} \omega\left(s-1; f; p^{\frac{1}{2}}\right)\left(1 + \vartheta_{i,s}\right),$$

where: $FF = d\left[F_{X_i}(x) + F_{Z_i}(x)\right]$.

Thus,

$$\left\|T_{S_\upsilon} f - T_Z f\right\| \le 2\frac{d^{\frac{s-1}{2}}}{(s-1)!} p^{\frac{s-1}{2}} WE,$$

$$WE = \omega\left(s-1; f; p^{\frac{1}{2}}\right) E\left[\sum_{i=1}^{\upsilon}\left(1+\vartheta_{i,s}\right)\right]$$

If $f^{(s-1)} \in Lip(\alpha)$, then

$$\left\|T_{S_\upsilon} f - T_Z f\right\| = O\left\{p^{\frac{s-1+\alpha}{2}} E\left[\sum_{i=1}^{\upsilon}\left(1+\vartheta_{i,s}\right)\right]\right\}.$$

Furthermore, if X_n are independent and identically distributed d-dimensional random variables, then

$$\left\|T_{S_\upsilon} f - T_Z f\right\| = O\left(p^{\frac{s-3+\alpha}{2}}\right).$$

The proof of the theorem is completed.

Theorem 2. Let $\{X_n, n \ge 1\}$ be a sequence of d-dimensional independent random variables for which

$$\vartheta_i(j) := \sum_{\substack{1\le i_1, i_2,\dots,i_j \le d \\ s_1+s_2+\dots+s_j=j}} \left|\int_{R^d} x_{i_1}^{s_1} x_{i_2}^{s_2} \dots x_{i_j}^{s_j} FF\right| = 0, \quad (6)$$

$$FF = d\left[F_{X_i}(x) - F_{Z_i}(x)\right].$$

and

$$\vartheta_{i,s} := \int_{R^d} \|x\|^s |FF| < +\infty, \quad (7)$$

$$FF = d\left[F_{X_i}(x) + F_{Z_i}(x)\right].$$

where:

$s \ge 2$ is a fixed integer, $1 \le j \le s, i = 1, 2, \dots, n$. Further, the following condition is held

$$\lim_{p\to 0} p^{\frac{s}{2}} \sum_{n=1}^{\infty} P(\nu = n) \sum_{i=1}^{n} \int_{\|x\| \ge \delta p^{\frac{-1}{2}}} \|x\|^s dF_{X_i}(x) = 0. \quad (8)$$

Then, for any $f \in C_B^s(R^d)$,

$$\left\|T_{S_\upsilon} f - T_Z f\right\| = o\left[p^{\frac{s}{2}} E\left(\sum_{i=1}^{\upsilon} \vartheta_{i,s}\right)\right] \quad as \quad p \to 0.$$

Further, if X_n are independent and identically distributed d-dimensional random variables, then

$$\left\|T_{S_\upsilon} f - T_Z f\right\| = o\left(p^{\frac{s-2}{2}}\right)(p \to 0).$$

Proof. For $f \in C_B^s(R^d)$, in view of the lemma 1, there is

$$\left|T_{p^{\frac{1}{2}} X_i} f(y) - \sum_{i=0}^{s} \frac{p^{\frac{j}{2}}}{j!} \int_{R^d} (x\nabla)^j f(y) dF_{X_i}(x)\right| \quad ,(9)$$

$$\le \frac{p^{\frac{s}{2}}}{s!} \int_{R^d} \left|(x\nabla)^s \left[f(\eta_1) - f(y)\right]\right| dF_{X_i}(x)$$

where: $\|\eta_1 - y\| \le p^{\frac{1}{2}} \|x\|$ and

$$\left|T_{p^{\frac{1}{2}} Z_i} f(y) - \sum_{i=0}^{s} \frac{p^{\frac{j}{2}}}{j!} \int_{R^d} (x\nabla)^j f(y) dF_{Z_i}(x)\right| \quad , (10)$$

$$\le \frac{p^{\frac{s}{2}}}{s!} \int_{R^d} \left|(x\nabla)^s \left[f(\eta_2) - f(y)\right]\right| dF_{Z_i}(x)$$

where: $\|\eta_2 - y\| \le p^{\frac{1}{2}} \|x\|$. On account of (4), (5) and lemma 3, combining (9), (10) and by an easy computation it follows that

$$\left|T_{p^{\frac{1}{2}} X_i} f(y) - T_{p^{\frac{1}{2}} Z_i} f(y)\right|$$

$$\le DPS\left\{\varepsilon.VF + 2\left\|f^{(s)}\right\| \int_{\|x\| \ge p^{\frac{-1}{2}}\delta} \|x\|^s dF_{X_i}(x)\right\},.$$

$$DPS = d^{\frac{s}{2}} \frac{p^{\frac{s}{2}}}{s!},$$

$$VF = \left(\vartheta_{i,s} + 2\left\|f^{(s)}\right\|\right).$$

Therefore,

$$\left\|T_{S_\upsilon} f - T_{S_{\nu^*}} f\right\|$$

$$\le \varepsilon.DPS \sum_{n=1}^{\infty} PVF + 2\left\|f^{(s)}\right\| DPS \sum_{n=1}^{\infty} PF$$

$$\le \varepsilon.\frac{d^{\frac{s}{2}}}{s!}\left\{p^{\frac{s}{2}} E\left[\sum_{i=1}^{\upsilon}\left(\vartheta_{i,s} + 2\left\|f^{(s)}\right\|\right)\right] + 2\left\|f^{(s)}\right\|\right\},$$

where:

$$DPS = \frac{d^{\frac{s}{2}}}{s!} p^{\frac{s}{2}},$$

$$\text{PVF} = P(\nu = n) \sum_{i=1}^{n} \left(\vartheta_{i,s} + 2 \left\| f^{(s)} \right\| \right),$$

$$PF = P(\nu = n) \sum_{i=1}^{n} \int_{\|x\| \geq p^{\frac{-1}{2}} \delta} \|x\|^{s} \, dF_{X_i}(x).$$

Thus,

$$\left\| T_{S_\upsilon} f - T_Z f \right\| = o\left\{ p^{\frac{s}{2}} E\left[\sum_{i=1}^{\upsilon} \left(\vartheta_{i,s} + 2 \left\| f^{(s)} \right\| \right) \right] \right\} \quad as \quad p \to 0.$$

Moreover, if X_n are independent and identically distributed d-dimensional random variables, then

$$\left\| T_{S_\upsilon} f - T_Z f \right\| = o\left(p^{\frac{s-2}{2}} \right) (p \to 0).$$

The proof is complete finished.

The results of this paper are extensions and generalizations of studies published in [10], [11] and [12].

Conclusions

Thus, the main results of the paper are presented by theorem 1 and theorem 2. The rate of convergence of geometric random sums to Laplace random variable on d-dimensional space is established. In particular, the first theorem gives us the convergence rate type of large-O. The convergence rate type of small-o is given in the second theorem. The results of the research can be useful in assessing the rate of convergence of approximations in such information technology as machine learning. These results will be more interesting and valuable if we discuss a rate of convergence of geometric random sums in the case of dependent random variables. The authors shall continue studying this matter in our future research.

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БАГАТОВИМІРНА АПРОКСИМАЦІЯ ЛАПЛАСА ІЗ ЗАСТОСУВАННЯМ ОПЕРАТОРА ТРОТТЕРА

Анотація. *Класичне розподілення Лапласа поряд з нормальним, стало однією з найбільш активно використовуваних симетричних імовірнісних моделей. Окремою задачею математики є апроксимація Лапласа, тобто спосіб оцінки параметрів нормального розподілення при апроксимації заданої щільності ймовірності. В даній статті досліджено задачу апроксимації Лапласа в d-вимірному просторі. Зокрема, вивчені швидкості збіжності в задачах багатовимірної апроксимації Лапласа. Математичним засобом, використаним в даній статті, є операційний метод, розроблений Троттером. Доведено дві теореми для оцінки швидкості збіжності. Швидкості збіжності, доведені в теоремах, виражаються за допомогою двох різних типів результатів, а саме: отримані оцінки швидкості збіжності апроксимації в термінах «О велике» і «о мале». Отримані результати можуть застосовуватися при використанні апроксимації Лапласа в задачах машинного навчання. Результати статті представляють собою новий підхід до задач апроксимації Лапласа для d-мірних незалежних випадкових величин.*

Ключові слова: *апроксимація Лапласа; геометричні суми; випадкові суми; оператор Троттера; швидкість збіжності*

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МНОГОМЕРНАЯ АППРОКСИМАЦИЯ ЛАПЛАСА С ПРИМЕНЕНИЕМ ОПЕРАТОРА ТРОТТЕРА

Аннотация. *Классическое распределение Лапласа наряду с нормальным, стало одной из наиболее активно используемых симметричных вероятностных моделей. Отдельной задачей математики является аппроксимация Лапласа, т.е. способ оценки параметров нормального распределения при аппроксимации заданной плотности вероятности. В данной статье исследована задача аппроксимации Лапласа в d-размерном пространстве. В частности, изучены скорости сходимости в задачах многомерной аппроксимации Лапласа. Математическим средством, использованным в данной статье, является операторный метод, разработанный Троттером. Доказаны две теоремы для оценки схорости сходимости. Скорости сходимости, доказанные в теоремах, выражаются с помощью двух разных типов результатов, а именно: получены оценки скорости сходимости аппроксимации в терминах «О большое» и «о малое». Полученные результаты могут применяться при использовании аппроксимации Лапласа в задачах машинного обучения. Результаты статьи представляют собой новый подход к задачам аппроксимации Лапласа для d-мерных независимых случайных величин.*

Ключевые слова: *аппроксимация Лапласа; геометрические суммы; случайные суммы; оператор Троттера; скорость сходимости*