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Design of modified Duffing pendulum for trajectory generation

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ABSTRACT

Relevance. The article proposes a new approach to the construction and analysis of chaotic dynamical systems based on distance-based state variables. In contrast to traditional state-space models, where the state of the system is described directly by coordinate variables, the proposed method represents the system using distances between the current state and a set of predefined reference (base) points. These base points can remain fixed in space or move along given trajectories. Such a geometric representation of the system state offers additional flexibility for modeling nonlinear dynamics and allows the construction of new types of chaotic behavior. **Purpose and objectives**. In this framework, the evolution of the system is described in terms of distances to selected base points, rather than through initial coordinates. This transformation generalizes traditional state-space descriptions and allows the developer to influence the structure of the dynamics by choosing the number, configuration, and movement of the reference points. As a result, the proposed formulation provides a convenient way to introduce additional nonlinear interactions into the system while maintaining a clear geometric interpretation. **Methods**. To demonstrate the proposed approach, the Duffing pendulum is used as a representative example. The classical Duffing oscillator is a well-known nonlinear system that can exhibit periodic, quasi-periodic, and chaotic oscillations. In this work, the Duffing dynamics is reformulated using distance-based variables defined with respect to several reference points. Two scenarios are considered: systems with fixed reference points and systems where the reference points move in time. In the latter case, the motion of the reference points can be periodic or chaotic, which further enriches the behavior of the system as a whole. Both continuous and discrete formulations of the model are developed. The continuous representation allows for the analysis of theoretical properties of the system, while the discrete version is well suited for numerical simulation and implementation on digital platforms and embedded devices. **Results**. Numerical experiments show that the modified systems reproduce the main qualitative features of the classical Duffing oscillator, as well as create new types of attractors. In particular, the models can generate unipolar and more complex multidimensional chaotic attractors. **Conclusions**. These results indicate that distance-based observability equations can serve as an effective tool for the design of new chaotic systems. The proposed approach can be useful in applications such as secure communication, parallel generation of chaotic signals, and the study of complex nonlinear dynamics.

Keywords: Duffing pendulum; chaotic systems; distance-based state variables; nonlinear observability; multi-dimensional attractors

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1. INTRODUCTION

The rapid development of unmanned aerial vehicles (UAVs) and advanced air transportation technologies has significantly increased the demand for reliable trajectory planning algorithms [1], [2], [3]. Modern UAV systems are expected to operate autonomously in complex environments that may include dense urban infrastructure, uncertain meteorological conditions, and dynamically changing obstacles [4], [5], [6]. In such scenarios, trajectory planning algorithms must ensure not only feasibility of the flight path but also robustness, adaptability, and safety under unpredictable disturbances [7], [8], [9].

The importance of this problem continues to grow as UAVs become widely used in applications such as environmental monitoring, logistics, emergency response, surveillance, and urban air mobility. In many of these missions, UAVs must navigate through environments that are highly constrained and subject to uncertain external influences such as turbulent wind fields, sensor noise, and unmodeled aerodynamic effects.

Consequently, trajectory planning methods must be capable of generating flexible flight paths that can adapt to disturbances while satisfying strict safety and operational constraints. Most existing trajectory planning techniques rely on deterministic optimization and classical control theory.

In such approaches, the trajectory is typically obtained by minimizing a predefined cost function subject to kinematic and dynamic constraints. These methods perform well in structured environments with predictable dynamics; however, their effectiveness may decrease in highly dynamic and uncertain conditions. In particular, deterministic planners may struggle to maintain performance when the UAV encounters complex disturbances, rapidly changing obstacles, or incomplete environmental information [10], [11], [12]. To address these limitations, researchers have increasingly explored nonlinear dynamics and chaotic systems as alternative tools for trajectory generation [13], [14], [15].

Chaotic systems possess several properties that make them attractive for such applications. Although deterministic, they are capable of generating complex aperiodic trajectories with strong sensitivity to initial conditions. This property enables the creation of non-repetitive motion patterns that can improve obstacle avoidance and reduce trajectory predictability in complex environments [16], [17], [18].

Classical nonlinear oscillators such as Duffing oscillators and Chua’s circuits have been investigated as potential generators of chaotic trajectories. These systems can produce rich dynamical behavior using relatively simple mathematical models and have been successfully applied to synchronization problems and inverse dynamic control tasks [20].

In addition, modern embedded computing platforms, including microcontrollers (MCUs), digital signal processors (DSPs), and field-programmable gate arrays (FPGAs), make it possible to implement chaotic dynamical systems directly in digital form [21], [22], [23]. Such implementations support both fixed-point and floating-point arithmetic and enable real-time computation of nonlinear dynamics for UAV control applications.

Despite these advances, most existing studies rely on a limited set of classical chaotic oscillators whose structure was originally developed for theoretical analysis rather than for trajectory generation problems. As a result, the design of chaotic trajectory generators remains largely constrained by the properties of these predefined systems.

Systematic methods for constructing new chaotic dynamical models that can be adapted to specific trajectory planning tasks are still relatively underdeveloped.

For example, the use of geometric relations between system states and spatial reference points for constructing chaotic systems has received little attention. Such geometric formulations could potentially provide additional flexibility for trajectory generation and allow chaotic dynamics to be tailored to the spatial constraints typical of UAV missions.

Therefore, a research gap exists in the development of constructive approaches that enable the systematic synthesis of chaotic dynamical systems specifically designed for trajectory generation and compatible with digital implementation platforms. To address this gap, this paper proposes a new framework for constructing chaotic systems based on distance-based state variables and geometric relations between system trajectories and reference points. This formulation allows classical nonlinear oscillators to be extended and transformed into trajectory generators with controllable dynamical properties and to generate new classes of chaotic attractors suitable for trajectory planning applications.

The main contributions of this work can be summarized as follows:

- a geometric framework for constructing chaotic dynamical systems using distance-based state variables and reference points is proposed;
- a formulation of second-order dynamical systems in triangular coordinates is developed and generalized to arbitrary dynamical systems defined in generalized phase-space representations;
- discrete-time models suitable for numerical simulation and digital implementation on embedded platforms are derived;
- new chaotic systems based on the Duffing oscillator are constructed for the cases of both fixed and moving reference points;

- the dynamical properties of the proposed systems are investigated through numerical simulations, demonstrating the emergence of new types of chaotic attractors that can be used for trajectory generation.

To the best of our knowledge, the use of distance-based state representations for systematic construction of chaotic dynamical systems intended for UAV trajectory generation has not been previously investigated in the literature.

The proposed approach therefore provides a new perspective on the design of chaotic trajectory generators and opens additional possibilities for the synthesis of complex nonlinear motion patterns in autonomous aerial systems.

2. LITERATURE REVIEW AND PROBLEM STATEMENT

Trajectory planning for unmanned aerial vehicles has been widely studied in recent decades, and a large number of approaches have been proposed to address the challenges associated with autonomous navigation in complex environments. Most classical methods rely on deterministic optimization and control theory, where the trajectory is generated by minimizing a predefined cost function subject to kinematic and dynamic constraints [24], [25], [26]. Such approaches typically provide reliable solutions in structured environments with well-defined constraints.

However, their performance may degrade when UAVs operate in highly dynamic or uncertain conditions, where disturbances and environmental changes cannot be accurately predicted in advance.

One important research direction focuses on probabilistic and stochastic trajectory planning methods. These approaches incorporate uncertainty directly into the planning process by modeling environmental disturbances and sensor noise using probabilistic frameworks [27], [28]. Probabilistic planners can improve robustness by accounting for uncertain obstacle positions or atmospheric disturbances. Nevertheless, such methods often require accurate statistical models of the environment and may become computationally demanding when the dimensionality of the planning problem increases.

Another active area of research involves the application of machine learning and data-driven techniques to UAV trajectory generation. Neural networks, reinforcement learning algorithms, and predictive models can assist in estimating environmental states and selecting appropriate control actions [29], [30], [31].

These approaches can significantly improve adaptability and enable UAV systems to operate autonomously in previously unseen environments. However, machine learning methods often require large training datasets and may exhibit limited interpretability. In addition, ensuring safety and stability guarantees for data-driven trajectory generators remains a challenging problem.

In parallel with these developments, nonlinear dynamics and chaotic systems have been investigated as alternative tools for trajectory generation and motion control [32], [33], [34]. Chaotic systems possess several properties that make them attractive for trajectory planning. They are deterministic but capable of producing complex aperiodic motion with strong sensitivity to initial conditions. As a result, chaotic trajectories can provide natural variability and unpredictability, which may improve obstacle avoidance and reduce trajectory predictability in hostile environments [35], [36], [37].

Several classical nonlinear oscillators, including Duffing oscillators and Chua's circuits, have been explored as generators of chaotic motion patterns. These systems allow complex trajectories to be generated using relatively simple mathematical models. For instance, interval-based Duffing oscillators have been used to generate chaotic trajectories and to solve inverse dynamic problems associated with UAV control [38].

Despite these promising results, most existing studies focus on the use of predefined chaotic systems rather than on systematic methods for constructing new chaotic models tailored to specific trajectory planning tasks.

Another emerging direction involves hybrid methods that combine chaotic dynamics with optimization algorithms. In these approaches chaotic systems generate candidate trajectories, while metaheuristic optimization techniques refine them to satisfy mission constraints. Examples include particle swarm optimization, whale optimization algorithms, and other population-based search methods [39], [40], [41]. Although these hybrid techniques can improve trajectory diversity and exploration capability, they often increase computational complexity and may require careful parameter tuning. At the same time, practical implementation issues must also be considered.

Modern UAV control systems typically rely on embedded digital platforms such as microcontrollers, digital signal processors, and field-programmable gate arrays [42], [43], [44]. These platforms allow nonlinear dynamical models to be implemented directly in digital form, which simplifies the integration of chaotic trajectory generators into real-time control systems. However, many existing chaotic models were originally designed for theoretical analysis or analog circuit implementations and may not be well suited for efficient digital realization.

Despite the considerable progress achieved in the areas discussed above, several important challenges remain unresolved. In particular, there is still a lack of systematic methods for constructing chaotic dynamical systems specifically tailored for trajectory generation problems. Most studies rely on a limited set of well-known chaotic oscillators, and the possibility of designing new chaotic models with controllable geometric properties has received relatively little attention. Furthermore, existing approaches often do not explicitly exploit geometric relationships between the system state and spatial reference points, which could provide additional flexibility for trajectory generation in constrained environments.

3. RESEARCH AIM AND OBJECTIVES

The aim of this research is to develop and investigate a novel approach for constructing chaotic dynamical systems suitable for trajectory generation in unmanned aerial vehicles operating in complex and uncertain environments. The proposed approach is based on distance-based representations of system states and geometric relationships between the system trajectory and a set of reference points.

Such a formulation is intended to provide a flexible framework for generating complex and non-repetitive trajectories while remaining compatible with digital implementation on modern computational platforms used in UAV control systems.

The research is also aimed at demonstrating that chaotic systems constructed using the proposed methodology can extend classical nonlinear models and produce a wider variety of attractor structures that may be useful for trajectory planning tasks.

In particular, the study focuses on the modification of well-known nonlinear oscillators and their transformation into trajectory generators capable of operating under mission-specific constraints and environmental disturbances.

To achieve this aim, the following objectives are addressed:

1) to analyze existing approaches to UAV trajectory planning based on nonlinear and chaotic dynamical systems;

2) to develop a mathematical framework for constructing chaotic dynamical systems using distance-based state variables and geometric relations between system states and reference points;

3) to formulate dynamical models of second-order systems in triangular coordinates and extend these models to generalized phase-space representations;

4) to derive discrete-time formulations of the proposed systems suitable for numerical simulation and digital implementation on embedded platforms;

5) to construct chaotic trajectory generators based on the Duffing oscillator for both fixed and moving reference points;

6) to investigate the dynamical properties of the proposed systems and analyze the resulting attractor structures using numerical simulations.

4. METHODOLOGY

4.1. The Generalized Second-Order Dynamical System in Triangular Coordinates on the System Phase Plane

We consider a generalized second-order controllable time-dependent dynamical system described by

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2, u_1, t), \\ \dot{y}_2 &= f_2(y_1, y_2, u_2, t), \quad y_i(0) = y_{i0}, \end{aligned} \tag{1}$$

where t represents the system operating time, y_1 and y_2 are state variables with initial values y_{i0}, u_1 and u_2 are control inputs, and $f_1(\cdot)$, $f_2(\cdot)$ define the system structure.

The solutions of (1) can be studied in several ways. The standard approach is to integrate the system over time to obtain the trajectories $y_i(t)$. Such signals can be generated and utilized in applications requiring dynamic response analysis or signal synthesis. However, because these trajectories are derived from nonlinear differential equations, tasks such as stability analysis or trajectory design can become challenging. Understanding these solutions is crucial for designing systems with specific dynamical properties.

A common technique is to divide the first equation by the second, yielding the phase-plane representation

$$\frac{dy_1}{dy_2} = \frac{f_1(y_1, y_2, u_1, t)}{f_2(y_1, y_2, u_2, t)}. \tag{2}$$

Unlike (1), this equation defines the system’s phase portrait directly in the (y_1, y_2) plane, without the need for time integration (Fig. 1).

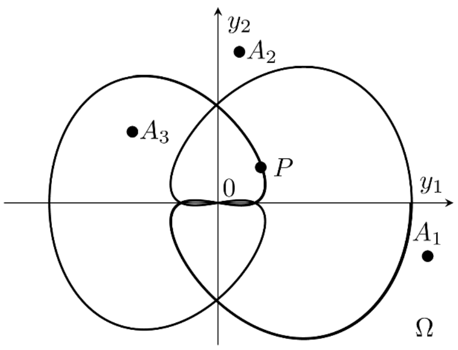

Fig 1. **Example of the system's phase portrait**
Source: **compiled by the authors**

We define three base points $A_i = (y_{1i}, y_{2i})$ in the plane Ω and denote a representative point $P = (y_1, y_2)$ corresponding to the system's current state. While Cartesian coordinates describe P using projections onto the axes, it is often convenient to represent its position by distances to the base points:

$$d_i^2 = (y_{1i} - y_1)^2 + (y_{2i} - y_2)^2, \quad i \in [1,3]. \tag{3}$$

From a control-theoretic perspective, (3) can be interpreted as nonlinear observability equations associated with (1). Combining (1) and (3) yields

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2, u_1, t), \\ \dot{y}_2 &= f_2(y_1, y_2, u_2, t), \quad y_j(0) = y_{j0},\ j \in [1,2], \\ d_i^2 &= (y_{1i} - y_1)^2 + (y_{2i} - y_2)^2, \quad i \in [1,3]. \end{aligned} \tag{4}$$

Accurate determination of the representative point requires the distances to all three base points. This approach allows the generation of more output variables than the number of state variables. Adding a new base point A_j introduces a new output d_j, with the restriction

$$y_{1j}^2 + y_{2j}^2 \neq y_{1i}^2 + y_{2i}^2, \quad i \in [0,n], \tag{5}$$

where n is the number of existing base points.

We claim that there are no any rules to select the number and position of base points from theoretical viewpoint excluding strong requirement that base points must not be placed in the same line. We believe that points' selection in random way increases system secured features. At the same time, from practical viewpoint triangle or other more complex figure which is defined by the base points should not have small angles between figure's sides to obtain correct calculation results caused by finite precession of calculation units.

Increasing the number of outputs enables the construction of a multichannel system that can generate a variety of motion trajectories based on nonlinear combinations of state variables and base-point coordinates. These trajectories can be computed by solving (1) and substituting the results into (3), making the method suitable for implementation on MCU, FPGA, or SoC platforms.

A limitation of this approach is that it results in differential-algebraic equations for the system dynamics, which require specialized analysis methods. To overcome this, we consider the inverse dynamic problem, expressing the motion directly in terms of distances to the base points.

Solving (3) for the coordinates of the representative point gives

$$\begin{aligned} y_1 &= f_{11} d_1^2 + f_{12} d_2^2 + f_{13} d_3^2 + f_1; \\ y_2 &= f_{21} d_1^2 + f_{22} d_2^2 + f_{23} d_3^2 + f_2, \end{aligned} \tag{6}$$

$$\begin{aligned} &a_0 = 2\big((y_{13} - y_{12}) y_{21} + (y_{11} - y_{13}) y_{22} + (y_{12} - y_{11}) y_{23}\big); \\ &a_1 = (y_{22} - y_{23}) y_{21}^2 + (y_{13}^2 - y_{12}^2 - y_{22}^2 + y_{23}^2) y_{21} + \\ &+ y_{22}^2 y_{23} + (y_{11}^2 - y_{13}^2 - y_{23}^2) y_{22} + y_{23}(y_{12}^2 - y_{11}^2); \\ &a_2 = (y_{13} - y_{12}) y_{11}^2 + (y_{12}^2 - y_{13}^2 + y_{22}^2 - y_{23}^2) y_{11} - \\ &- y_{12}^2 y_{13} + (y_{13}^2 - y_{21}^2 + y_{23}^2) y_{12} - y_{13}(y_{22}^2 - y_{21}^2); \\ &b_{11} = y_{23} - y_{22};{}_{12} = y_{21} - y_{23}; b_{13} = y_{22} - y_{21}; \\ &b_{21} = y_{12} - y_{13};{}_{22} = y_{13} - y_{11}; b_{33} = y_{11} - y_{12}; \\ &f_{11} = b_{11} / a_0;\ f_{12} = b_{12} / a_0;\ f_{13} = b_{13} / a_0;\ f_1 = a_1 / a_0; \\ &f_{21} = b_{21} / a_0;\ f_{22} = b_{22} / a_0;\ f_{23} = b_{23} / a_0;\ f_2 = a_2 / a_0; \end{aligned} \tag{7}$$

Substituting (6) into (1) gives

$$\begin{aligned} &g_{11}(\dot{d}_1, \dot{d}_2, \dot{d}_3, y_{ij}) = g_{12}(d_1, d_2, d_3, u_1, t, y_{ij}), \\ &g_{21}(\dot{d}_1, \dot{d}_2, \dot{d}_3, y_{ij}) = g_{22}(d_1, d_2, d_3, u_2, t, y_{ij}), \\ &d_i(0) = d_{i0}, \quad i, j \in [1,3]. \end{aligned} \tag{8}$$

These equations can be rewritten in standard first-order form:

$$\begin{aligned} \dot{d}_1 &= h_1(d_1, d_2, d_3, u_1, u_2, t, y_{ij}), \\ \dot{d}_2 &= h_2(d_1, d_2, d_3, u_1, u_2, t, y_{ij}), \\ \dot{d}_3 &= h_3(d_1, d_2, d_3, u_1, u_2, t, y_{ij}), \end{aligned} \tag{9}$$

where $h_i(\cdot)$ are nonlinear functions.

One way to define $h_i(\cdot)$ is to minimize the norm

$$I = \frac{1}{2} \|\dot{\mathbf{d}}\|^2 \to \min, \quad \dot{\mathbf{d}} = \begin{pmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{pmatrix}, \tag{10}$$

subject to the constraints

$$\begin{aligned} c_1 &= g_{11}(\dot{\mathbf{d}}, y_{ij}) - g_{12}(\mathbf{d}, u_1, t, y_{ij}) = 0, \\ c_2 &= g_{21}(\dot{\mathbf{d}}, y_{ij}) - g_{22}(\mathbf{d}, u_2, t, y_{ij}) = 0. \end{aligned} \tag{11}$$

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The Lagrange equations

$$\nabla_{\mathbf{d}} \mathrm{L} = \mathbf{d} + \lambda_1 \nabla c_1(\mathbf{d}) + \lambda_2 \nabla c_2(\mathbf{d}),$$
$$\mathrm{L}(\mathbf{d}, \lambda_1, \lambda_2) = \frac{1}{2}\mathbf{d}^T\mathbf{d} + \lambda_1 h_1(\mathbf{d}) + \lambda_2 h_2(\mathbf{d}), \quad (12)$$

with λ_i as the i-th Lagrange multiplier, allow computing $\dot{d}_i$ and determining the functions $h_i(\cdot)$.

Equations (9) thus describe the motion of the generalized dynamical system in terms of distances to the base points A_i. The order of this system depends on the number of base points and can exceed that of the original system.

4.2. The Generalized *n*-th Order Dynamical System in M-Angular Coordinates

The approach proposed for multichannel dynamical system design can be easily generalized to an n-th order system:

$$\dot{\mathbf{Y}} = \mathbf{F}(\mathbf{Y}, \mathbf{U}, t), \quad (13)$$

where the system state vector is

$$\mathbf{Y} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}^T, \quad (14)$$

the control vector is

$$\mathbf{U} = \begin{pmatrix} u_1 & u_2 & \cdots & u_m \end{pmatrix}^T, \quad (15)$$

and the vector function is

$$\mathbf{F}(\cdot) = \begin{pmatrix} f_1(\cdot) & f_2(\cdot) & \cdots & f_n(\cdot) \end{pmatrix}^T, \quad (16)$$

with f_i denoting the i-th component of the system vector function.

Assume that m base points are defined in the n-dimensional system phase space. The distances between each base point and the representative point can then be expressed as

$$\| d_j \|^2 = \sum_{i=1}^{n} (y_{ij} - y_i)^2, \quad i \in [1, n],\ j \in [1, m]. \quad (17)$$

Solving (13) for y_i allows the generalization of (6) in the form

$$\mathbf{Y} = \mathbf{W}(\mathbf{d}, \mathbf{Y_j}), \quad y_i = w_i(d_j, y_{ij}), \quad (18)$$

where $w_i(\cdot)$ are the components of an n-dimensional vector function depending on the vector of base point coordinates $\mathbf{Y_j}$ and the distance vector $\mathbf{d}$ to the representative point.

Substituting (18) into (13) yields the following m-dimensional system of equations:

$$\dot{\mathbf{d}} = \mathbf{H}(\mathbf{d}, \mathbf{Y_j}, \mathbf{U}, t). \quad (19)$$

Equation (19) describes the system motion when the base points are fixed. If the base points are moving, it is necessary to differentiate both arguments of the function $\mathbf{W}(\cdot)$ in (18), while the derivatives of the components of $\mathbf{Y}$ are defined.

This leads to the generalized form

$$\dot{\mathbf{d}} = \mathbf{H_v}(\mathbf{d}, \mathbf{Y_j}, \dot{\mathbf{Y}}_{\mathbf{j}}, \mathbf{U}, t), \quad (20)$$

where the subscript indicates variable (moving) base point coordinates.

Thus, equations (19) and (20) can be considered as the generalized motion equations of a dynamical system in an -angular coordinate framework, corresponding to fixed and moving base points, respectively.

Compared to known chaotic trajectory generators based on predefined oscillators [9-11], the proposed approach provides additional flexibility by allowing the structure of the system to be modified through geometric relations. Unlike optimization-based or learning-based planners, the method does not require iterative training or solving complex optimization problems, which makes it computationally efficient and suitable for real-time applications.

4.3. Discrete-Time Implementation of the Generalized *n*-th Order Dynamical System

The motion equations presented above are defined in the continuous-time domain, allowing the modeling of the underlying physical processes used in system design. However, implementing such systems in hardware typically requires complex analog devices, such as operational amplifiers, multipliers, and other components, to perform continuous-time transformations and calculations. Due to the inherent schematic complexity of these analog implementations, modern systems often rely on digital platforms, such as MCU-, FPGA-, or DSP-based boards, which can reproduce complex system dynamics with minimal external circuitry.

To facilitate digital implementation, we adapt (19) and (20) to the discrete-time domain by approximating continuous-time derivatives using finite differences. A wide range of numerical methods exists for this purpose; in this work, we consider a general approach that uses the current value of the differentiated state variable along with its previous values:

$$\frac{d}{dt} \approx D\left(1, z^{-1}, ..., z^{-q}, T_s\right) \quad (21)$$

where z^{-i} is the backward shift operator, and T_s is the sampling interval.

Substituting (21) into (19) and (20) yields

$$D(1,z^{-1},\ldots,z^{-q},T_s)\mathbf{d}=\mathbf{H}(\mathbf{d},\mathbf{Y_j},\mathbf{U},t), \quad (22)$$

$$D(1,z^{-1},\ldots,z^{-q},T_s)\mathbf{d}=\mathbf{H_v}(\mathbf{d},\mathbf{Y_j},\dot{\mathbf{Y}}_\mathbf{j},\mathbf{U},t), \quad (23)$$

which represent discrete-time approximations of the original continuous-time systems.

Using these discrete-time derivative approximations, equations (2) and (3) can be rewritten as recurrent (iterative) equations, allowing the computation of the current state based on previous system values:

$$\mathbf{d}=\mathbf{H_z}(z^{-j},\mathbf{d},\mathbf{Y_j},\mathbf{U},t,T_s), \quad (24)$$

$$\mathbf{d}=\mathbf{H_{vz}}(z^{-j},\mathbf{d},\mathbf{Y_j},\mathbf{U},t,T_s), \quad (25)$$

where the operator z^{-j} indicates the use of previous values of the system state variables.

Equations (24) and (25) provide a straightforward method for implementing the designed dynamical system on a variety of digital platforms.

5. RESEARCH RESULTS

5.1. Differential-Algebraic Duffing Pendulum Model

5.1.1. Duffing Pendulum Model with Fixed Base Points

We apply the proposed approach to design a novel chaotic system based on the well-known Duffing equation:

$$\begin{aligned}&\ddot{y}_1=-\delta\dot{y}_1-\alpha y_1-\beta y_1^3+\gamma\cos(\omega t),\\&y_1(0)=y_{10},\,\dot{y}_1(0)=y_{20}.\end{aligned} \quad (26)$$

The system is simulated with the parameters $\delta=0.02$, $\alpha=1$, $\beta=5$, $\gamma=8$, $\omega=0.5$, $y_{10}=1$, and $y_{20}=0$.

The choice of parameters such as α, β, and the sampling step ΔT significantly affects the system dynamics. In particular, parameter α controls the stiffness of the system, while β determines the degree of nonlinearity. The sampling step ΔT influences numerical stability and resolution of the generated trajectories. The selected values are parameters of classical well-studied Duffing pendulum which exhibits stable chaotic behavior and clear attractor structures. At the same time, one can use other parameters as well but this using should be proven by considering pendulum's bifurcation diagrams and studying its Lyapunov exponents as well as other chaotic indicators to prove the chaotic nature of such modified system.

The simulation results are shown in Fig. 2.

For the simulation, we employ the simplest finite-difference approximation for derivatives using backward differences:

$$\frac{d}{dt}\approx\frac{1-z^{-1}}{z^{-1}T_s},\quad \frac{d^2}{dt^2}\approx\frac{1-2z^{-1}+z^{-2}}{z^{-2}T_s^2}, \quad (27)$$

where z^{-1} is the backward shift operator and T_s is the sampling period.

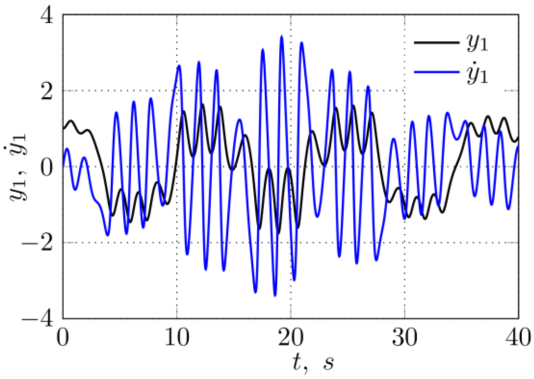

Fig. 2. **Duffing pendulum motion**
Source: **compiled by the authors**

Fig. 3 shows the pendulum phase portrait, including the representative point P and the three base points $A_i=(y_{1i},y_{2i})$. The coordinates of the base points are chosen as $y_{11}=-1$, $y_{21}=4$, $y_{12}=1.5$, $y_{22}=-3$, $y_{13}=-0.5$, $y_{23}=3.5$.

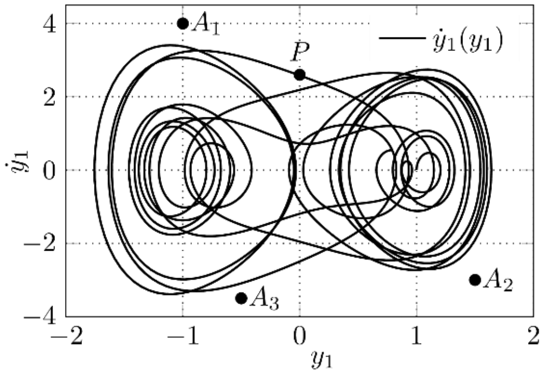

Fig. 3. **Duffing pendulum attractor**
Source: **compiled by the authors**

Based on these base points, the nonlinear observability equations are defined as

$$\begin{aligned}&d_1=\pm\sqrt{(y_1+1)^2+(\dot{y}_1-4)^2},\\&d_2=\pm\sqrt{(y_1-1.5)^2+(\dot{y}_1+3)^2},\\&d_3=\pm\sqrt{(y_1+0.4)^2+(\dot{y}_1+3.5)^2}.\end{aligned} \quad (28)$$

Using these observability equations, the system trajectories and the corresponding attractor are illustrated in Fig. 4 and Fig. 5.

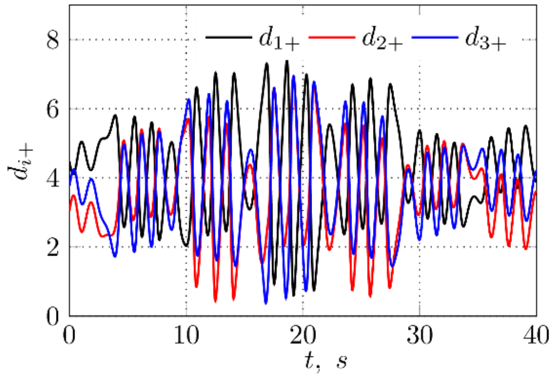

Fig. 4. **Duffing pendulum observed outputs**
Source: **compiled by the authors**

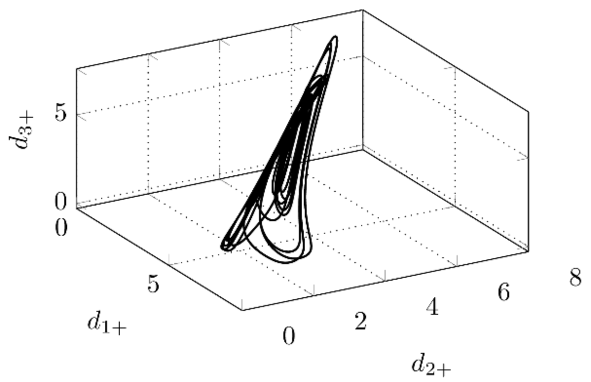

Fig. 5. **Modified Duffing pendulum 3D attractor**
Source: **compiled by the authors**

Analysis of Fig. 3 indicates that representing the Duffing pendulum with respect to selected base points generates oscillations that are different from the original ones but retain a similar waveform. Notably, the resulting signals are uni-polar instead of bipolar. The use of the square-root function in defining the system outputs enables the generation of uni-polar signals with a desired polarity, which can be advantageous for communication system applications. These observations demonstrate that the proposed approach allows the design of novel dynamical systems with additional features using nonlinear observability equations.

An important feature of the proposed approach is the ability to increase the number of system outputs. This makes it possible to visualize the system trajectory in three-dimensional space (Fig. 4) instead of the conventional two-dimensional plane (Fig. 2). Such 3D representations can facilitate the planning of chaotic motions and other applications that exploit parallel chaotic signal generation.

5.1.2. Duffing Pendulum Model with Moving Base Points

We now generalize (8) for the case of variable base point coordinates. In general, base points can move along arbitrary trajectories; in this study, we consider only periodic motions, which ensures that the distances d_i remain bounded.

First, we examine the Duffing pendulum with base points undergoing regular harmonic motions, described by the second-order differential equations:

$$\begin{aligned} &\dot{x}_{1j} = x_{2j}; \\ &\dot{x}_{2j} = -\omega_i \,|x_{1j}|^{\alpha_i}\, sign(x_{1j}), \quad x_{ij}(0) = y_{ij}, \end{aligned} \tag{29}$$

where ω_i is the oscillation frequency of the i-th base point, and α_i is a power factor to generate nonlinear oscillations.

Simulation results for the Duffing pendulum with periodically moving base points are shown in Fig. 6 and Fig. 7, obtained using $\omega_i = 3+i$ and $\alpha_i = 0.1i$, $i \in [1,3]$.

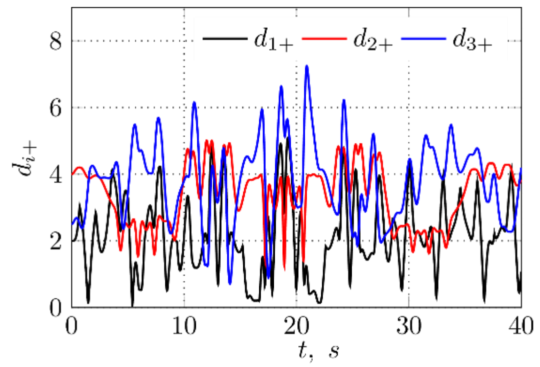

Fig. 6. **Oscillations in the Duffing pendulum with moving base points**
Source: **compiled by the authors**

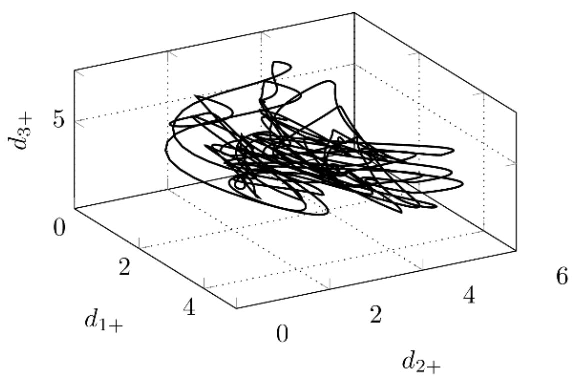

Fig. 7. **Modified Duffing pendulum 3D attractor for moving base points**
Source: **compiled by the authors**

As illustrated in Fig. 6, the motion of the base points increases the complexity of the system outputs, producing chaotic oscillations distinct from those of the conventional Duffing pendulum.

Next, we consider base points moving chaotically. The chaotic motion is generated using the following system:

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + b x_2 x_3^2, \quad x_1(0) = 1, \\ \dot{x}_2 &= c x_1 + d x_1 x_3^2, \quad x_2(0) = -1, \\ \dot{x}_3 &= h x_3 + k x_1^2, \quad x_3(0) = -3, \end{aligned} \tag{30}$$

with parameters $a = 10$, $b = 1$, $c = 5$, $d = -1$, $h = -5$, $k = -6$.

In this case, the distances to the base points in the system phase plane are defined as

$$d_i = \pm\sqrt{(y_1 - x_1)^2 + (\dot{y}_1 - \dot{x}_1)^2}. \tag{31}$$

Simulation results for the Duffing pendulum with chaotically moving base points are shown in Fig. 8 and Fig.9.

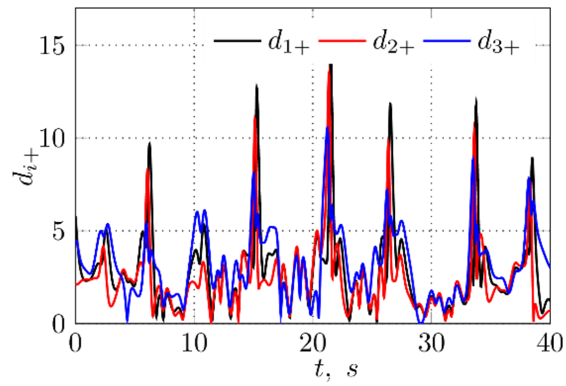

Fig. 8. **Oscillations in the Duffing pendulum with chaotically moving base points**
Source: **compiled by the authors**

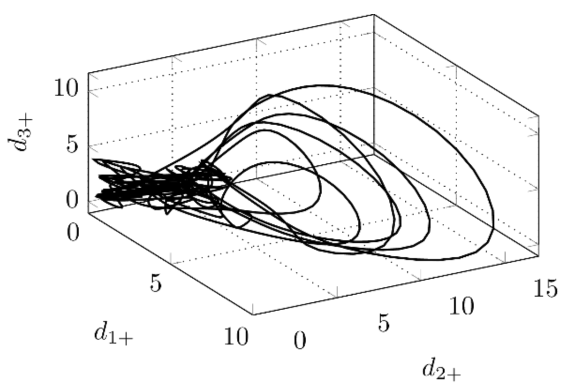

Fig. 9. **Modified Duffing pendulum 3D attractor with chaotically moving base points**
Source: **compiled by the authors**

Comparing Fig. 8, Fig. 6 and Fig. 4 demonstrates a significant difference from the initial system dynamics shown in Fig. 2. This confirms that the proposed approach enables the design of novel chaotic systems through appropriate mathematical transformations of well-known models. Furthermore, it allows combining several known systems – whether regular or chaotic – to generate new chaotic oscillations, provided at least one subsystem exhibits chaotic behavior.

The simulations above define distances between the representative point and base points. Since these distances are computed using the square-root function, each output has a symmetric counterpart with opposite sign.

The concept can be generalized by considering squared distances as new state variables, rewriting (28) as

$$\begin{aligned} d2_1 &= (y_1 + 1)^2 + (\dot{y}_1 - 4)^2, \\ d2_2 &= (y_1 - 1.5)^2 + (\dot{y}_1 + 3)^2, \\ d2_3 &= (y_1 + 0.4)^2 + (\dot{y}_1 + 3.5)^2. \end{aligned} \tag{32}$$

Simulation results using these squared distances are presented in Fig. 10, Fig. 11, Fig. 12, Fig. 13, Fig. 14 and Fig. 15, corresponding to the original results in Fig. 4, Fig. 5, Fig. 6, Fig. 7, Fig. 8 and Fig. 9.

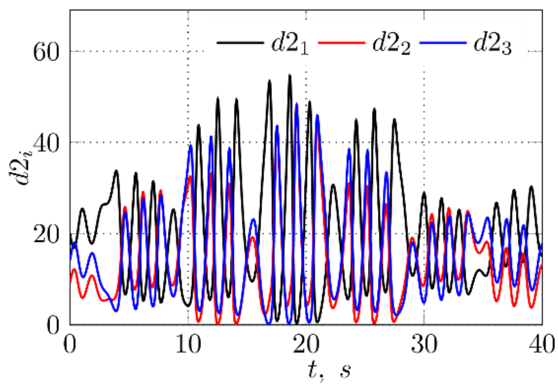

Fig. 10. **Duffing pendulum with observed squared distances outputs**
Source: **compiled by the authors**

As shown in Fig. 10, Fig.11, Fig.12, Fig.13, Fig.14 and Fig. 15, using observability equations modifies the system output trajectories, enabling the selection of a coordinate system that produces the desired motions. This coordinate system may be either Euclidean or non-Euclidean, and its choice clearly defines the observability equations as the equations used to calculate distances between points in the chosen coordinate system.

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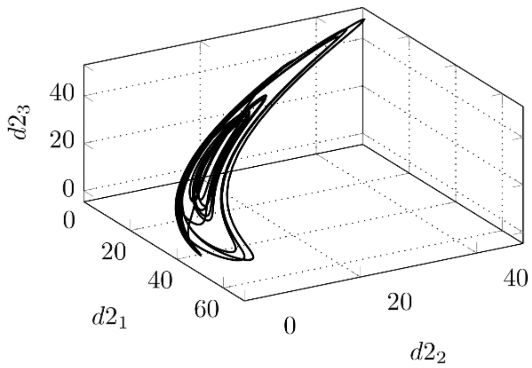

Fig 11. **Modified Duffing pendulum 3D attractor with squared distances outputs**
Source: **compiled by the authors**

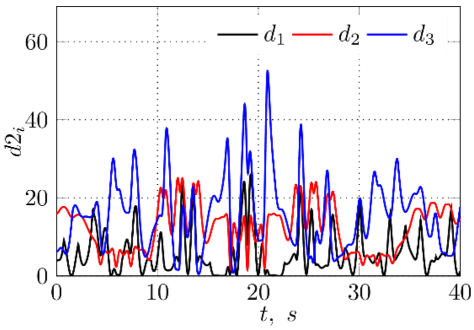

Fig. 12. **Oscillations with squared distances outputs and moving base points**
Source: **compiled by the authors**

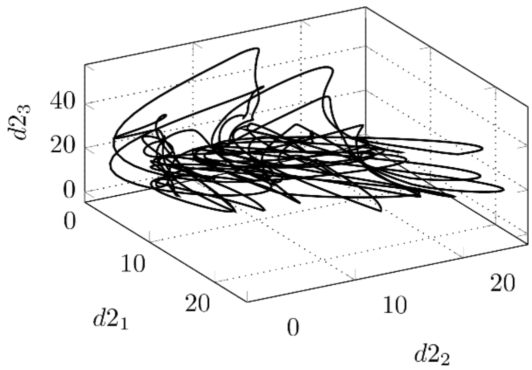

Fig. 13. **3D attractor for squared distances outputs with moving base points**
Source: **compiled by the authors**

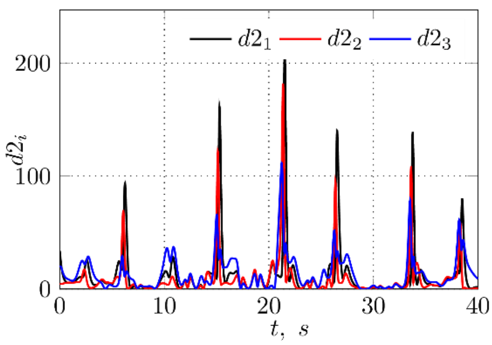

Fig. 14. **Oscillations with squared distances outputs and chaotically moving base points**
Source: **compiled by the authors**

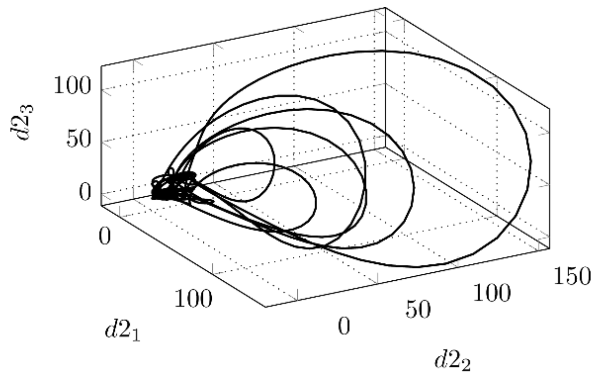

Fig. 15. **3D attractor for squared distances outputs with chaotically moving base points**
Source: **compiled by the authors**

If one assumes distances from representative point to base points as new cartesian coordinates, he can consider the designed in such a way system's attractors as motion trajectories for some UAV or other vehicles. The main feature of the proposed approach is increasing number of system outputs without increasing system order. Thus, nevertheless of defining system motion in 3D space, the order of considered system still equals two and the designed attractors are flat figures in 3D space. These figures have their own initial shift, angular orientation and forbidden zones. One can use this features to plan UAV motions in complex environment by choosing attractor which is located in free space from buildings or other obstacles.

5.2. Differential Duffing Pendulum Model in Terms of Distances to Base Points

From a control theory perspective, the chaotic systems presented above are constructed using the concept of state-space equations augmented with nonlinear observability equations. Mathematically, this approach relies on a combination of algebraic and differential equations. Its technical implementation is straightforward, allowing both fixed- and floating-point calculations on digital devices to generate the chaotic oscillations described previously.

However, employing this approach can introduce certain challenges when using the resulting differential-algebraic equations to address problems involving direct or inverse dynamics. Therefore, in this section, we focus on designing chaotic systems that treat distances as state variables directly, without the need to rely on observability equations.

5.2.1. Case of Fixed Base Points

Assuming that the base points remain stationary, and substituting (6) into (26), one obtains the following equation:

$$\begin{aligned}&2f_{21}d_1\dot{d}_1+2f_{22}d_2\dot{d}_2+2f_{23}d_3\dot{d}_3=\\&-(\alpha f_{11}+\delta f_{21})d_1^2-(\alpha f_{12}+\delta f_{22})d_2^2-\\&-(\alpha f_{13}+\delta f_{23})d_3^2-\alpha f_1-\delta f_2-\\&-\beta(f_{11}d_1^2+f_{12}d_2^2+f_{13}d_3^2+f_1)^3+\gamma\cos(\omega t).\end{aligned}\tag{33}$$

This equation is derived using the weight factors defined in (7), which are determined from the known coordinates of the base points.

We can interpret (33) as a sum of component equations and split it symmetrically as follows:

$$\begin{aligned}&2f_{21}d_1\dot{d}_1=-(\alpha f_{11}+\delta f_{21})d_1^2-(\alpha f_1-\delta f_2)/3-\\&-\beta(f_{11}d_1^2+f_{12}d_2^2+f_{13}d_3^2+f_1)^3/3+\gamma\cos(\omega t)/3;\\&2f_{22}d_2\dot{d}_2=-(\alpha f_{12}+\delta f_{22})d_2^2-(\alpha f_1-\delta f_2)/3-\\&-\beta(f_{11}d_1^2+f_{12}d_2^2+f_{13}d_3^2+f_1)^3/3+\gamma\cos(\omega t)/3;\\&2f_{23}d_3\dot{d}_3=-(\alpha f_{13}+\delta f_{23})d_3^2-(\alpha f_1-\delta f_2)/3-\\&-\beta(f_{11}d_1^2+f_{12}d_2^2+f_{13}d_3^2+f_1)^3/3+\gamma\cos(\omega t)/3.\end{aligned}\tag{34}$$

Other ways of splitting these equations are also possible; however, the chosen symmetric form allows the first three equations to be considered as having separable variables, while the remaining three can be treated as second-kind Abel equations.

Numerical integration of (34) is straightforward. By introducing the substitution

$$d2_i=d_i^2,\tag{35}$$

the system can be rewritten in terms of the new variables as

$$\begin{aligned}&f_{21}\dot{d}2_1=-(\alpha f_{11}+\delta f_{21})d2_1-(\alpha f_1-\delta f_2)/3-\\&-\beta(f_{11}d2_1+f_{12}d2_2+f_{13}d2_3+f_1)^3/3+\gamma\cos(\omega t)/3;\\&f_{22}\dot{d}2_2=-(\alpha f_{12}+\delta f_{22})d2_2-(\alpha f_1-\delta f_2)/3-\\&-\beta(f_{11}d2_1+f_{12}d2_2+f_{13}d2_3+f_1)^3/3+\gamma\cos(\omega t)/3;\\&f_{23}\dot{d}2_3=-(\alpha f_{13}+\delta f_{23})d2_3-(\alpha f_1-\delta f_2)/3-\\&-\beta(f_{11}d2_1+f_{12}d2_2+f_{13}d2_3+f_1)^3/3+\gamma\cos(\omega t)/3.\end{aligned}\tag{36}$$

Applying the backward-difference operator (7) to (36) yields a system of nonlinear finite-difference equations:

$$\begin{aligned}&f_{21}\frac{1-z^{-1}}{T_s}d2_1=-(\alpha f_{11}+\delta f_{21})z^{-1}d2_1-\\&-\beta(f_{11}z^{-1}d2_1+f_{12}z^{-1}d2_2+f_{13}z^{-1}d2_3+f_1)^3/3+\\&-(\alpha f_1-\delta f_2)/3+\gamma\cos(\omega(t-T_s))/3;\\&f_{22}\frac{1-z^{-1}}{T_s}d2_2=-(\alpha f_{12}+\delta f_{22})z^{-1}d2_2-\\&-\beta(f_{11}z^{-1}d2_1+f_{12}z^{-1}d2_2+f_{13}z^{-1}d2_3+f_1)^3/3+\\&-(\alpha f_1-\delta f_2)/3+\gamma\cos(\omega(t-T_s))/3;\\&f_{23}\frac{1-z^{-1}}{T_s}d2_3=-(\alpha f_{13}+\delta f_{23})z^{-1}d2_3-\\&-\beta(f_{11}z^{-1}d2_1+f_{12}z^{-1}d2_2+f_{13}z^{-1}d2_3+f_1)^3/3\\&-(\alpha f_1-\delta f_2)/3+\gamma\cos(\omega(t-T_s))/3.\end{aligned}\tag{37}$$

These equations can be solved for the current values of the new state variables $d2_i$ as

$$\begin{aligned}&f_{21}d2_1=-(T_s\alpha f_{11}+(T_s\delta+1)f_{21})z^{-1}d2_1-\\&-T_s(\alpha f_1-\delta f_2)/3+T_s\gamma\cos(\omega(t-T_s))/3-\\&-T_s\beta(f_{11}z^{-1}d2_1+f_{12}z^{-1}d2_2+f_{13}z^{-1}d2_3+f_1)^3/3;\\&f_{22}d2_2=-(T_s\alpha f_{12}+(T_s\delta+1)f_{22})z^{-1}d2_2-\\&-T_s(\alpha f_1-\delta f_2)/3+T_s\gamma\cos(\omega(t-T_s))/3-\\&-T_s\beta(f_{11}z^{-1}d2_1+f_{12}z^{-1}d2_2+f_{13}z^{-1}d2_3+f_1)^3/3;\\&f_{23}d2_3=-(T_s\alpha f_{13}+(T_s\delta+1)f_{23})z^{-1}d2_3-\\&-T_s(\alpha f_1-\delta f_2)/3+T_s\gamma\cos(\omega(t-T_s))/3-\\&-T_s\beta(f_{11}z^{-1}d2_1+f_{12}z^{-1}d2_2+f_{13}z^{-1}d2_3+f_1)^3/3.\end{aligned}\tag{38}$$

Equation (38) represents the discrete-time model of the Duffing pendulum in terms of squared distances to fixed base points. Numerical solutions of this system reproduce trajectories similar to those shown in Fig. 10, Fig.11, Fig.12, Fig.13, Fig. 14 and Fig. 15 Inverting the substitution (35),

$$d_i=\sqrt{d2_i},\tag{39}$$

recovers curves analogous to those in Fig. 4, Fig 5, Fig .6, Fig. 7, Fig. 8 and Fig. 9.

It is evident from (36) that this approach increases the effective order of the dynamical system and allows the coordinates of the system's representative point to be determined solely from its distances to the fixed base points.

5.2.2. Case of Moved Base Points

In the most general scenario, both the representative point and the base points may move in the phase plane. Such motions can arise from the physical processes described by the underlying

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mathematical models or may be introduced deliberately to enhance the system's security features.

In this context, the weight factors f_{ij} in (7) must be generalized to time-dependent functions that depend not only on the Duffing pendulum dynamics (26), but also on the regular (29) and chaotic (30) motions of the base points.

Substituting (6) into (26) now produces more complex equations due to the necessity of differentiating the time-dependent f_{ij} functions. Accordingly, equation (1) can be written in the following form:

$$\begin{aligned}&2f_{21}d_1\dot{d}_1+\dot{f}_{21}d_1^2+2f_{22}d_2\dot{d}_2+\\&+\dot{f}_{22}d_2^2+2f_{23}d_3\dot{d}_3+\dot{f}_{23}d_3^2=\\&=-(\alpha f_{11}+\delta f_{21})d_1^2-(\alpha f_{12}+\delta f_{22})d_2^2-\\&-(\alpha f_{13}+\delta f_{23})d_3^2-\alpha f_1-\delta f_2-\\&-\beta(f_{11}d_1^2+f_{12}d_2^2+f_{13}d_3^2+f_1)^3+\gamma\cos(\omega t).\end{aligned} \tag{40}$$

Unlike (1), equation (8) contains additional terms of the form $\dot{f}_{2i}d_i^2$, which depend not only on the positions of the base points but also on the velocities of their motions. This allows us to rewrite the symmetric splitting (34) as

$$\begin{aligned}&2f_{21}d_1\dot{d}_1=-(\alpha f_{11}+\delta f_{21}+\dot{f}_{21})d_1^2-(\alpha f_1-\delta f_2)/3-\\&-\beta(f_{11}d_1^2+f_{12}d_2^2+f_{13}d_3^2+f_1)^3/3+\gamma\cos(\omega t)/3;\\&2f_{22}d_2\dot{d}_2=-(\alpha f_{12}+\delta f_{22}+\dot{f}_{22})d_2^2-(\alpha f_1-\delta f_2)/3-\\&-\beta(f_{11}d_1^2+f_{12}d_2^2+f_{13}d_3^2+f_1)^3/3+\gamma\cos(\omega t)/3;\\&2f_{23}d_3\dot{d}_3=-(\alpha f_{13}+\delta f_{23}+\dot{f}_{23})d_3^2-(\alpha f_1-\delta f_2)/3-\\&-\beta(f_{11}d_1^2+f_{12}d_2^2+f_{13}d_3^2+f_1)^3/3+\gamma\cos(\omega t)/3.\end{aligned} \tag{41}$$

Equation (41) represents the Duffing pendulum expressed in terms of distances d_i to moving base points.

By applying the substitution (3), the system can be rewritten in terms of squared distances $d2_i$ as

$$\begin{aligned}&2f_{21}\dot{d}2_1=-(\alpha f_{11}+\delta f_{21}+\dot{f}_{21})d2_1-(\alpha f_1-\delta f_2)/3-\\&-\beta(f_{11}d2_1+f_{12}d2_2+f_{13}d2_3+f_1)^3/3+\gamma\cos(\omega t)/3;\\&2f_{22}\dot{d}2_2=-(\alpha f_{12}+\delta f_{22}+\dot{f}_{22})d2_2-(\alpha f_1-\delta f_2)/3-\\&-\beta(f_{11}d2_1+f_{12}d2_2+f_{13}d2_3+f_1)^3/3+\gamma\cos(\omega t)/3;\\&2f_{23}\dot{d}2_3=-(\alpha f_{13}+\delta f_{23}+\dot{f}_{23})d2_3-(\alpha f_1-\delta f_2)/3-\\&-\beta(f_{11}d2_1+f_{12}d2_2+f_{13}d2_3+f_1)^3/3+\gamma\cos(\omega t)/3.\end{aligned} \tag{10}$$

Equation (42) describes the Duffing pendulum in terms of squared distances to moving base points. Applying standard numerical approximations for derivative operators, along with the explicit expressions for the f_{ij} functions and the trajectories of the base points, allows the construction of discrete-time models of the system. Due to their complexity, these discrete-time formulations are not presented here.

It should be noted that alternative models can also be formulated by adopting different definitions of distances between points.

6. DISCUSSION OF RESULTS

The obtained analytical expressions and numerical simulation results demonstrate that the proposed approach provides a structured and formalized framework for constructing chaotic dynamical systems. The use of geometric relations and distance-based state variables makes it possible to represent system dynamics in a compact mathematical form while preserving a clear physical interpretation of the system motion.

Unlike traditional formulations where system dynamics are directly expressed in terms of Cartesian coordinates, the proposed approach introduces distance-based variables defined with respect to a set of reference points. Such a representation allows the dynamical model to incorporate geometric information about the system motion and enables the construction of new classes of nonlinear systems derived from known oscillatory models.

The simulation results obtained for the Duffing-based systems demonstrate that the proposed framework is capable of generating a wide variety of dynamical regimes. In particular, both periodic and chaotic attractors were observed depending on the configuration of the reference points and the parameters of the underlying nonlinear oscillator.

The introduction of moving reference points significantly enriches the system dynamics and leads to the formation of more complex attractor structures. An important advantage of the proposed method is the possibility of constructing chaotic systems in a systematic way by modifying known dynamical models. Instead of relying on a limited set of classical chaotic oscillators, the developed framework allows new chaotic systems to be synthesized by introducing geometric relations between system states and reference points. This provides additional flexibility when designing trajectory generators for specific applications.

Another advantage of the proposed approach is its compatibility with digital implementation. The derived discrete-time models can be efficiently implemented on modern computational platforms such as microcontrollers, digital signal processors, or field-programmable gate arrays. This makes the proposed chaotic systems suitable for real-time trajectory generation and control in UAV systems.

The general algorithm for constructing such dynamical systems can be summarized as follows.

1. A base nonlinear dynamical system describing the motion of a physical or abstract oscillator is selected.
2. A set of reference points is introduced in the state space. These points can be either fixed or moving according to predefined trajectories.
3. Distance-based state variables are defined as geometric relations between the system state and the reference points.
4. The original system equations are transformed into a new representation expressed in terms of these distance variables.
5. The resulting dynamical system is analyzed analytically and numerically to determine its stability properties and attractor structures.
6. A discrete-time version of the system is constructed to enable numerical simulation and digital implementation.

The numerical experiments confirm that the proposed approach is capable of generating both classical and previously unexplored attractor structures. In particular, the obtained results demonstrate the emergence of uni-polar attractors and complex multidimensional chaotic trajectories that may be useful for trajectory planning applications.

At the same time, several practical aspects should be considered when applying the proposed framework. For instance, the choice of reference point configuration and system parameters has a significant influence on the resulting system dynamics.

Therefore, further research is required to develop systematic parameter selection strategies for specific trajectory planning tasks.

Nevertheless, the results indicate that the proposed distance-based formulation provides a promising tool for constructing chaotic dynamical systems and generating complex motion patterns.

Such systems may be particularly useful in applications that require flexible and non-repetitive trajectories, including autonomous navigation of UAVs in cluttered or uncertain environments.

Overall, the developed approach expands the class of available chaotic models and provides additional opportunities for the design of nonlinear trajectory generators in modern autonomous systems.

We see the following future research directions in development of adaptive synchronization mechanisms for multi-agent systems using interval-based chaotic oscillators to improve safety and mission success rates.

7. CONCLUSIONS

Thus, in the paper is proposed a novel approach to the design and analysis of chaotic dynamical systems, using the Duffing pendulum as a representative example.

The key findings and contributions can be highlighted as follows.

1. We introduced a distance-based formulation of dynamical systems by defining system state variables in terms of distances to fixed or moving base points. This approach generalizes conventional state-space modeling and allows for the construction of new chaotic behaviors.
2. For the Duffing pendulum, both fixed and moving base points were considered. Fixed base points enable the generation of uni-polar outputs while preserving the overall structure of the original oscillations. Moving base points, including periodically and chaotically moving ones, produce richer and more complex chaotic dynamics, demonstrating the flexibility of the proposed method.
3. We derived both continuous- and discrete-time formulations of the modified Duffing pendulum in terms of distances and squared distances. The discrete-time models facilitate numerical implementation on digital platforms such as MCU-, FPGA-, or DSP-based systems.
4. The proposed framework allows the combination of multiple known dynamical systems, including systems with regular and chaotic motion, to synthesize novel chaotic oscillations. This capability opens possibilities for applications in secure communications, parallel signal generation, and complex motion planning.
5. Numerical simulations confirmed that the modified systems reproduce and extend the dynamics of the classical Duffing pendulum. The use of distance-based observability equations or squared-distance variables provides an effective means to control system trajectories and generate desired chaotic patterns.
6. Overall, the presented approach offers a general and flexible methodology for designing chaotic systems with tunable dynamics and observable outputs, providing a foundation for further studies in nonlinear control, chaos engineering, and applied dynamics.

8. ACKNOWLEDGMENTS

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АНОТАЦІЯ

Актуальність. У статті пропонується новий підхід до побудови та аналізу хаотичних динамічних систем на основі змінних стану, що базуються на відстані. На відміну від традиційних моделей простору станів, де стан системи описується безпосередньо координатними змінними, запропонований метод представляє систему з використанням відстаней між поточним станом та набором попередньо визначених опорних (базових) точок. Ці базові точки можуть залишатися фіксованими в просторі або рухатися за заданими траєкторіями. Таке геометричне представлення стану системи пропонує додаткову гнучкість для моделювання нелінійної динаміки та дозволяє конструювати нові типи хаотичної поведінки. **Мета і завдання.** У цій структурі еволюція системи описується через відстані до вибраних базових точок, а не через вихідні координати. Це перетворення узагальнює традиційні описи простору станів і дозволяє розробнику впливати на структуру динаміки, вибираючи кількість, конфігурацію та рух опорних точок. Як результат, запропонована формулювання забезпечує зручний спосіб введення додаткових нелінійних взаємодій у систему, зберігаючи при цьому чітку геометричну інтерпретацію. **Методи.** Для демонстрації запропонованого підходу як репрезентативний приклад використовується маятник Даффінга. Класичний осцилятор Даффінга — це добре відома нелінійна система, яка може демонструвати періодичні, квазіперіодичні та хаотичні коливання. У цій роботі динаміка Даффінга переформульована з використанням змінних на основі відстані, визначених відносно кількох базових точок. Розглядаються два сценарії: системи з фіксованими базовими точками та системи, де базові точки рухаються в часі. В останньому випадку рух опорних точок може бути періодичним або хаотичним, що ще більше збагачує поведінку системи в цілому. Розроблено як неперервні, так і дискретні формулювання моделі. Неперервне представлення дозволяє аналізувати теоретичні властивості системи, тоді як дискретна версія добре підходить для числового моделювання та реалізації на цифрових платформах та вбудованих пристроях. **Результати**. Числові експерименти показують, що модифіковані системи відтворюють основні якісні особливості класичного осцилятора Даффінга, а також створюють нові типи атракторів. Зокрема, моделі можуть генерувати однополярні та складніші багатовимірні хаотичні атрактори. **Висновки**. Ці результати вказують на те, що рівняння спостережливості на основі відстані можуть служити ефективним інструментом для проектування нових хаотичних систем. Запропонований підхід може бути корисним у таких застосуваннях, як безпечний зв'язок, паралельна генерація хаотичних сигналів та вивчення складної нелінійної динаміки.

Ключові слова: маятник Даффінга; хаотичні системи; змінні стану на основі відстані; нелінійна спостережуваність; багатовимірні атрактори

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